

Description of the Heterotic String Solutions in the M Model

P. A. Bolokhov^{a,b}, M. Shifman^c, and A. Yung^{c,d}

^a*Physics and Astronomy Department, University of Pittsburgh, Pittsburgh,
Pennsylvania, 15260, USA*

^b*Theoretical Physics Department, St.Petersburg State University, Ulyanovskaya 1,
Peterhof, St.Petersburg, 198504, Russia*

^c*William I. Fine Theoretical Physics Institute, University of Minnesota,
Minneapolis, MN 55455, USA*

^d*Petersburg Nuclear Physics Institute, Gatchina, St. Petersburg 188300, Russia*

Abstract

We continue the study of heterotic non-Abelian BPS-saturated flux tubes (strings). Previously, such solutions were obtained in $U(N)$ gauge theories: $\mathcal{N} = 2$ supersymmetric QCD deformed by superpotential terms $\mu \mathcal{A}^2$ breaking $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$. In these models one cannot consider the limit $\mu \rightarrow \infty$ which would eliminate adjoint fields: the bulk theory develops a Higgs branch; the emergence of massless particles in the bulk precludes one from taking the limit $\mu \rightarrow \infty$. This drawback is absent in the M model (hep-th/0701040) where the matter sector includes additional “meson” fields M introduced in a special way. We generalize our previous results to the M model, derive the heterotic string (the string world-sheet theory is a heterotic $\mathcal{N} = (0, 2)$ sigma model, with the $CP(N-1)$ target space for bosonic fields and an extra right-handed fermion coupled to the fermion fields of the $\mathcal{N} = (2, 2)$ $CP(N-1)$ model), and then explicitly obtain all relevant zero modes. This allows us to relate parameters of the microscopic M model to those of the world-sheet theory. The limit $\mu \rightarrow \infty$ is perfectly smooth. Thus, the full-blown and fully analyzed heterotic string emerges, for the first time, in the $\mathcal{N} = 1$ theory with no adjoint fields. The fate of the confined monopoles is discussed.

1 Introduction

Non-Abelian BPS-saturated flux tubes were discovered [1, 2] and studied [3, 4, 5] in $\mathcal{N} = 2$ supersymmetric QCD with the gauge group $U(N)$, the Fayet–Iliopoulos (FI) term, and N flavors (N hypermultiplets in the fundamental representation), for reviews see [6, 7, 8, 9]. If $\mathcal{N} = 2$ supersymmetry is maintained in the bulk, the low-energy theory on the string world sheet is split into two disconnected parts: a free theory for (super)translational moduli and a nontrivial part, a theory of interacting (super)orientational moduli described by $\mathcal{N} = 2$ supersymmetric $CP(N - 1)$ sigma model. The above splitting of the moduli space is completely predetermined by the fact that the basic bulk theory has eight supercharges, and the string under consideration is 1/2 BPS (classically).

If $\mathcal{N} = 2$ bulk theory is deformed by mass terms $\mu\mathcal{A}^2$ of the adjoint fields, breaking $\mathcal{N} = 2$ down to $\mathcal{N} = 1$, the situation drastically changes: two of the four former supertranslational modes become coupled to two superiorientational modes [10]. As a result, the world sheet theory is deformed too. Instead of the well-studied $\mathcal{N} = (2, 2)$ $CP(N - 1)$ model we now have a heterotic $\mathcal{N} = (0, 2)$ sigma model, with the $CP(N - 1)$ target space for bosonic fields and an extra right-handed fermion which couples to the fermion fields of the $CP(N - 1)$ model in a special way. In the previous works [11, 12] the heterotic world-sheet model was derived from the microscopic theory by a direct calculation of all relevant zero modes. This allowed us to relate the heterotic $\mathcal{N} = (0, 2)$ sigma model parameters with those of the bulk theory.

The task we addressed was moving away from $\mathcal{N} = 2$, towards $\mathcal{N} = 1$. In particular, it is highly desirable to get rid of all adjoint fields inherent to $\mathcal{N} = 2$ models. If we were able to tend $\mu \rightarrow \infty$ this goal would be achieved, all adjoint fields would become infinitely heavy and could be eliminated. Unfortunately, simultaneously with increasing the masses of the adjoint fields the bulk theory develops a Higgs branch, and massless (light) moduli fields come with it. The string swells, and all approximations fail at $\mu > \mu_*$ where μ_* is a critical value,

$$\mu_* \sim \frac{\xi}{\Lambda_\sigma}, \quad (1.1)$$

ξ is the FI coefficient and Λ_σ is the dynamical scale of the world-sheet sigma model. Although μ_* can be made large, there are crucial questions which cannot be addressed under the constraint $\mu \ll \mu_*$. One of them is the fate of the kinks in the heterotic $CP(N - 1)$ model which, from the bulk standpoint, represent confined monopoles. At $\mu \ll \mu_*$ the world-sheet theory has N degenerate (albeit quantum-mechanically nonsupersymmetric) vacua which are well defined. Correspondingly, the kink masses

are well-defined too; in fact, they were calculated [13] in the large- N limit. In a formal limit $\mu \rightarrow \infty$ the above degenerate vacua coalesce. Physics of the kinks becomes obscure.

To avoid this problem an M model was designed [14]. Besides the fields present in the $\mathcal{N} = 1$ deformation of the basic $\mathcal{N} = 2$ bulk theory, the M model includes N^2 “mesonic” superfields, which break $\mathcal{N} = 2$ right from the start. The M model is characterized by one extra interaction constant h . It was demonstrated [14] that at finite h the limit $\mu \rightarrow \infty$ becomes smooth. Therefore, one can completely eliminate the adjoint fields. The solitonic flux tube solution (BPS-saturated at the classical level) persists. Our task in this paper is to derive the world-sheet theory for these strings (in the limit of small h and $\mu \rightarrow \infty$). We prove that it is the same heterotic $\text{CP}(N-1)$ model, with specific relations between constants of this model and those of the bulk theory. To obtain these relations we determine all relevant zero modes for the M -model flux-tube solution.

The paper is organized as follows. In Sect. 2 we review M model, string solutions and the bosonic part of the world-sheet theory on the non-Abelian string. In Sect. 3 we calculate fermionic supertranslational and superorientational modes of the string and derive the fermionic part of the world-sheet theory. Our derivation shows that the world-sheet theory is the heterotic $\mathcal{N} = (0, 2)$ supersymmetric $\text{CP}(N-1)$ model. In Sect. 4 we discuss the fate of the bulk monopoles confined on the string in the limit of large μ . Section 5 contains our brief conclusions. Our notation is summarized in Appendix of Ref. [12].

2 The M Model

In this section we describe how non-Abelian strings emerge in the M Model. Our discussion here parallels that of [14] where more details are given, and we review only the most essential points. The theory we start with is $\mathcal{N} = 2$ Supersymmetric Quantum Chromodynamics (SQCD) with the gauge group $\text{SU}(N) \times \text{U}(1)$ and $\mathcal{N} = 2$ supersymmetry explicitly broken to $\mathcal{N} = 1$ by the following deformations:

$$\delta\mathcal{W}_{\mathcal{N}=1} = \sqrt{2N}\mu_1 (\mathcal{A}^{\text{U}(1)})^2 + \frac{\mu_2}{2} (\mathcal{A}^a)^2 + \text{Tr } M \tilde{Q} Q. \quad (2.1)$$

Here the first two terms break supersymmetry by giving masses to the adjoint supermultiplets $\mathcal{A}^{\text{U}(1)}$ and $\mathcal{A}^{\text{SU}(N)}$, while M breaks supersymmetry by coupling to the quark fields. In fact, M is the superfield extension of the quark mass matrix m_A^B . It promotes m_A^B to a dynamical chiral superfield,

$$\delta S_{M\text{kin}} = \int d^4x d^2\theta d^2\bar{\theta} \frac{2}{h} \text{Tr } \bar{M} M,$$

where h is a dimensionless coupling constant.

Why do we introduce $\mu\mathcal{A}^2$ deformations? The reason behind the introduction of the masses μ_1 and μ_2 is the desire to make the adjoint fields heavy and exclude them from low-energy physics. The role of the superfield M is to lift the Higgs branch which appears when the adjoints are integrated out. If the coupling constant $h = 0$, the M field is frozen at an arbitrary constant and returns to its original status of the quark mass matrix. This does not break $\mathcal{N} = 2$ supersymmetry. However, once $h \neq 0$, the coupling to M becomes a deformation which breaks $\mathcal{N} = 2$ supersymmetry down to $\mathcal{N} = 1$ (in conjunction with nonvanishing parameters μ_1 and μ_2).

The bosonic part of the theory is

$$\begin{aligned}
S_{\text{bos}} = \int d^4x \left(\frac{1}{2g_2^2} \text{Tr} (F_{\mu\nu}^{\text{SU}(N)})^2 + \frac{1}{g_1^2} (F_{\mu\nu}^{\text{U}(1)})^2 + \right. \\
\frac{2}{g_2^2} \text{Tr} |\nabla_\mu a^{\text{SU}(N)}|^2 + \frac{4}{g_1^2} |\partial_\mu a^{\text{U}(1)}|^2 + |\nabla_\mu q^A|^2 + |\nabla_\mu \tilde{q}^A|^2 + \\
\left. \frac{1}{h} |\partial_\mu M^0|^2 + \frac{1}{h} |\partial_\mu M^a|^2 + V(q^A, \tilde{q}_A, a^a, a^{\text{U}(1)}, M^0, M^a) \right). \tag{2.2}
\end{aligned}$$

Here ∇_μ is the covariant derivative in the appropriate representation

$$\begin{aligned}
\nabla_\mu^{\text{adj}} &= \partial_\mu - i [A_\mu^a T^a, \cdot], \\
\nabla_\mu^{\text{fund}} &= \partial_\mu - i A_\mu^{\text{U}(1)} - i A_\mu^a T^a.
\end{aligned}$$

The vector fields A_μ and complex scalars a belong to gauge multiplets of the $\text{U}(1)$ and $\text{SU}(N)$ sectors, respectively, while q^{kA} and \tilde{q}_{Ak} denote squarks, $k = 1, \dots, N$ and $A = 1, \dots, N_f$ are the color and flavor indices, respectively. In this paper we consider only the case $N_f = N$.

The matrix superfield M_B^A is conveniently decomposed as

$$M_B^A = \frac{1}{2} \delta_B^A M^0 + (T^a)_B^A M^a.$$

Assembling results of Refs. [11, 14] (see also [8]) one can readily see that the potential

of the theory (2.2) takes the form

$$\begin{aligned}
V(q^A, \tilde{q}_A, a^a, a^{U(1)}, M^0, M^a) &= \\
&= \frac{g_2^2}{2} \left(\frac{1}{g_2^2} f^{abc} \bar{a}^b a^c + \text{Tr} \bar{q} T^a q - \text{Tr} \tilde{q} T^a \tilde{\bar{q}} \right)^2 \\
&+ \frac{g_1^2}{8} (\text{Tr} \bar{q} q - \text{Tr} \tilde{\bar{q}} \tilde{q} - N\xi)^2 \\
&+ 2g_2^2 \left| \text{Tr} \tilde{q} T^a q + \frac{1}{\sqrt{2}} \frac{\partial \mathcal{W}_{\mathcal{N}=1}}{\partial a^a} \right|^2 + \frac{g_1^2}{2} \left| \text{Tr} \tilde{q} q + \frac{1}{\sqrt{2}} \frac{\partial \mathcal{W}_{\mathcal{N}=1}}{\partial a^{U(1)}} \right|^2 \\
&+ 2 \text{Tr} \left\{ \left| (a^{U(1)} + a^a T^a) q + \frac{1}{\sqrt{2}} q \left(\frac{M^0}{2} + M^a T^a \right) \right|^2 \right. \\
&\quad \left. + \left| (a^{U(1)} + a^a T^a) \tilde{\bar{q}} + \frac{1}{\sqrt{2}} \tilde{\bar{q}} \left(\frac{M^0}{2} + M^a T^a \right) \right|^2 \right\} \\
&+ \frac{h}{4} |\text{Tr} \tilde{q} q|^2 + h |\text{Tr} q T^a \tilde{q}|^2 .
\end{aligned} \tag{2.3}$$

This potential is a sum of F and D terms, in particular the last two terms in (2.3) are the F terms of the M field. We also introduced the Fayet–Iliopoulos D term in the third line, with the real (and positive) parameter ξ . The $\mathcal{N} = 2$ supersymmetry is broken by parameters μ_1 , μ_2 and h via $\mathcal{W}_{\mathcal{N}=1}$, while the FI term does not break $\mathcal{N} = 2$ supersymmetry [15, 16].

A nonvanishing ξ in the potential triggers condensation of quarks and spontaneous breaking of the gauge symmetry. The vacuum expectation values (VEVs) of the quarks can be chosen in the form

$$\begin{aligned}
\langle q^{kA} \rangle &= \sqrt{\xi} \begin{pmatrix} 1 & 0 & \dots \\ \dots & \dots & \dots \\ \dots & 0 & 1 \end{pmatrix} , & \langle \tilde{\bar{q}}^{kA} \rangle &= 0 , \\
k &= 1, \dots, N , & A &= 1, \dots, N ,
\end{aligned} \tag{2.4}$$

i.e. the so-called color-flavor locked form. The adjoint VEVs have to vanish classically,

$$\langle a^{SU(N)} \rangle = 0 , \quad \langle a^{U(1)} \rangle = 0 , \tag{2.5}$$

together with the VEVs of the M field,

$$\langle M^a \rangle = 0 , \quad \langle M^0 \rangle = 0 . \tag{2.6}$$

Despite the full Higgsing of the gauge symmetry, the VEVs (2.4), (2.5), and (2.6) leave a global diagonal $SU(N)_{C+F}$ symmetry unbroken,

$$q \rightarrow UqU^{-1}, \quad a^{SU(N)} \rightarrow Ua^{SU(N)}U^{-1}, \quad M \rightarrow UMU^{-1}. \quad (2.7)$$

In what follows, we will be interested in the limit of very large μ_1, μ_2 . It appears that the VEV structure (2.4), (2.5) and (2.6) does not depend on the supersymmetry breaking parameters, owing to the fact that the adjoint fields vanish in the vacuum, see Eq. (2.5). In particular, the VEVs will retain the same form up to very large μ .

To allow the theory to be treated semiclassically, we arrange it to be at weak coupling, by separating the dynamical scale of $SU(N)$ from the scale of the gauge symmetry breaking ξ as follows:

$$\sqrt{\xi} \gg \Lambda_{SU(N)}.$$

The perturbative spectrum was discussed in detail in [14], and we will only concentrate on the limit of large μ . Regardless of μ , the gauge bosons acquire mass,

$$m_{\text{ph}} = g_1 \sqrt{\frac{N}{2}} \xi \quad (2.8)$$

for the $U(1)$ gauge boson (“photon”) and

$$m_W = g_2 \sqrt{\xi} \quad (2.9)$$

for the $SU(N)$ bosons.

The scalar bosons line up in the following hierarchy of scales. The heaviest bosons, in the $\mu_i \gg \sqrt{\xi}$ limit, have the masses

$$\begin{aligned} m_{U(1)}^{(\text{largest})} &= \sqrt{\frac{N}{2}} g_1^2 \mu_1, \\ m_{SU(N)}^{(\text{largest})} &= g_2^2 \mu_2, \end{aligned} \quad (2.10)$$

with the first mass carried by two degenerate states, while the second mass is carried by $2(N^2 - 1)$ states. These are the masses of heavy adjoint scalars $a^{U(1)}$ and $a^{SU(N)}$. The low-energy bulk theory spectrum consists of light states with the masses

$$\begin{aligned} m_{U(1)}^{(1)} &= \sqrt{\frac{hN\xi}{4}} \left\{ 1 + \frac{\sqrt{\xi}}{2g_1\mu_1} \sqrt{\gamma_1(\gamma_1 + 1)} + \dots \right\}, \\ m_{U(1)}^{(2)} &= \sqrt{\frac{hN\xi}{4}} \left\{ 1 - \frac{\sqrt{\xi}}{2g_1\mu_1} \sqrt{\gamma_1(\gamma_1 + 1)} + \dots \right\}, \end{aligned} \quad (2.11)$$

with two states for each, and of states with the masses

$$\begin{aligned} m_{\text{SU}(N)}^{(1)} &= \sqrt{\frac{h\xi}{2}} \left\{ 1 + \frac{\sqrt{\xi}}{2g_2\mu_2} \sqrt{\gamma_2(\gamma_2+1)} + \dots \right\}, \\ m_{\text{SU}(N)}^{(2)} &= \sqrt{\frac{h\xi}{2}} \left\{ 1 - \frac{\sqrt{\xi}}{2g_2\mu_2} \sqrt{\gamma_2(\gamma_2+1)} + \dots \right\}, \end{aligned} \quad (2.12)$$

with $2(N^2 - 1)$ degenerate states for each of the values.

At non-zero h there are no massless states in the bulk theory, even if $\mu_i \rightarrow \infty$. One can integrate out the heavy adjoint fields, obtaining

$$\begin{aligned} S_{\text{bos}}^{M \text{ model}} &= \int d^4x \left(\frac{1}{2g_2^2} \text{Tr} (F_{\mu\nu}^{\text{SU}(N)})^2 + \frac{1}{g_1^2} (F_{\mu\nu}^{\text{U}(1)})^2 + \text{Tr} |\nabla_\mu q|^2 + \text{Tr} |\nabla_\mu \tilde{q}|^2 \right. \\ &\quad + \frac{1}{h} |\partial_\mu M^0|^2 + \frac{1}{h} |\partial_\mu M^a|^2 + \frac{g_2^2}{2} (\text{Tr} \bar{q} T^a q - \text{Tr} \tilde{q} T^a \tilde{q})^2 \\ &\quad + \frac{g_1^2}{8} (\text{Tr} \bar{q} q - \text{Tr} \tilde{q} \tilde{q} - N\xi)^2 + \text{Tr} |q M|^2 + \text{Tr} |\tilde{q} M|^2 \\ &\quad \left. + \frac{h}{4} |\text{Tr} \tilde{q} q|^2 + h |\text{Tr} q T^a \tilde{q}|^2 \right). \end{aligned} \quad (2.13)$$

The vacuum of this theory is given in Eqs. (2.4) and (2.6). The perturbative excitations consist of the $\mathcal{N} = 1$ gauge multiplets with masses (2.8) and (2.9), and of chiral multiplets with masses determined by the leading terms in Eqs. (2.11) and (2.12). The scale of the $\mathcal{N} = 1$ theory is related to that of the original $\mathcal{N} = 2$ theory as follows:

$$\Lambda_{\mathcal{N}=1}^{2N} = \mu_2^N \Lambda_{\text{SU}(N)}^N. \quad (2.14)$$

By taking the FI parameter large enough, $g_2\sqrt{\xi} \gg \Lambda_{\mathcal{N}=1}$, we ensure the $\mathcal{N} = 1$ theory (the M model) is at weak coupling.

The theory (2.13) admits the existence of non-Abelian strings, the presence of which can be traced from the $\mathcal{N} = 2$ theory (2.2). A Z_N string can be written in

terms of the profile functions [2, 14]

$$\begin{aligned}
q &= \begin{pmatrix} \phi_2(r) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \phi_2(r) & 0 \\ 0 & 0 & \dots & e^{i\alpha}\phi_1(r) \end{pmatrix}, \quad \tilde{q} = 0, \\
A_i^{\text{SU}(N)} &= \frac{1}{N} \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} (\partial_i \alpha) (-1 + f_N(r)), \\
A_i^{\text{U}(1)} &= \frac{1}{N} (\partial_i \alpha) (1 - f(r)) \cdot \mathbf{1}, \quad A_0^{\text{U}(1)} = A_0^{\text{SU}(N)} = 0, \\
a^{\text{U}(1)} &= a^{\text{SU}(N)} = M^0 = M^a = 0,
\end{aligned} \tag{2.15}$$

where r and α are the polar coordinates in the plane orthogonal to the string, while the index $i = 1, 2$ labels the Cartesian coordinates in this plane. The quark profile functions $\phi_1(r)$, $\phi_2(r)$, and the gauge profile functions $f(r)$ and $f_N(r)$ obey a system of first-order differential equations

$$\begin{aligned}
\partial_r \phi_1(r) - \frac{1}{Nr} \left(f(r) + (N-1)f_N(r) \right) \phi_1(r) &= 0, \\
\partial_r \phi_2(r) - \frac{1}{Nr} \left(f(r) - f_N(r) \right) \phi_2(r) &= 0, \\
\partial_r f(r) - r \frac{Ng_1^2}{4} \left((N-1)\phi_2(r)^2 + \phi_1(r)^2 - N\xi \right) &= 0, \\
\partial_r f_N(r) - r \frac{g_2^2}{2} \left(\phi_1(r)^2 - \phi_2(r)^2 \right) &= 0,
\end{aligned} \tag{2.16}$$

with the boundary conditions

$$\begin{aligned}
\phi_1(0) &= 0, \quad \phi_2(0) \neq 0, \quad \phi_1(\infty) = \sqrt{\xi}, \quad \phi_2(\infty) = \sqrt{\xi}, \\
f_N(0) &= 1, \quad f(0) = 1, \quad f_N(\infty) = 0, \quad f(\infty) = 0.
\end{aligned} \tag{2.17}$$

The tension of the Z_N string (2.15) is

$$T_1 = 2\pi\xi.$$

Besides the position of the center of the string x_0 , a genuine non-Abelian string also possesses collective coordinates in the group space $\text{SU}(N)_{\text{C+F}}$, which determine the orientation of the string in the group. The solution (2.15) breaks $\text{SU}(N)_{\text{C+F}}$ down to $\text{SU}(N-1) \times \text{U}(1)$. Therefore, the space of the orientational coordinates is given by the coset

$$\frac{\text{SU}(N)}{\text{SU}(N-1) \times \text{U}(1)} \sim \text{CP}(N-1). \tag{2.18}$$

A general non-Abelian string solution can be obtained from Eqs. (2.15) by applying a $SU(N)_{C+F}$ rotation U , namely,

$$\begin{aligned}
q &= U \begin{pmatrix} \phi_2(r) & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \phi_2(r) & 0 \\ 0 & 0 & \dots & \phi_1(r) \end{pmatrix} U^{-1}, \quad \tilde{q} = 0, \\
A_i^{SU(N)} &= \frac{1}{N} U \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} U^{-1} (\partial_i \alpha) f_N(r), \quad (2.19) \\
A_i^{U(1)} &= -\frac{1}{N} (\partial_i \alpha) f(r) \cdot \mathbf{1}, \quad A_0^{U(1)} = A_0^{SU(N)} = 0, \\
a^{U(1)} &= a^{SU(N)} = M^0 = M^a = 0,
\end{aligned}$$

where we have passed to the singular gauge in which the quark field does not wind, while the gauge field winds around the origin.

The bosonic string solution (2.19) at the classical level does not involve \tilde{q} , the adjoint fields a , or the M fields; thus it is independent of the supersymmetry breaking parameters. In particular, this solution will retain its form when μ_2 is taken very large and the adjoints are integrated out. Therefore, this solution will still be present in the M model (2.13),

$$\begin{aligned}
q &= \phi_2 + n\bar{n}(\phi_1 - \phi_2), \\
A_i^{SU(N)} &= \varepsilon_{ij} \frac{x^i}{r^2} f_N(r) \left(n\bar{n} - 1/N \right), \quad (2.20) \\
A_i^{U(1)} &= \frac{1}{N} \varepsilon_{ij} \frac{x^i}{r^2} f(r), \\
\tilde{q} &= M^0 = M^a = 0.
\end{aligned}$$

We have introduced here the orientational collective coordinates n^l , which parametrize the rotation matrix U as follows

$$\frac{1}{N} U \begin{pmatrix} 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & -(N-1) \end{pmatrix} U^{-1} = -n^i \bar{n}_i + \frac{1}{N} \cdot \mathbf{1}_i, \quad (2.21)$$

where we deploy matrix notation on the left-hand side. The coordinates n^l ($l = 1, \dots, N$) form a complex vector in the fundamental representation of $SU(N)$ and live

in the $\text{CP}(N-1)$ space, *i.e.*

$$\bar{n}_l \cdot n^l = 1, \quad (2.22)$$

and one common complex phase of n^l can be gauged away (*e.g.* one can choose n^N to be real).

To obtain the effective sigma model on the string world sheet [2, 3, 17], one assumes the moduli n^l to be slowly-varying functions along the string, $n^l = n^l(x^k)$. Substituting then the solution (2.20) into the kinetic terms of the action (2.2), one arrives at the $\text{CP}(N-1)$ sigma model (see the review [8] for details)

$$S_{\text{CP}(N-1)}^{1+1} = 2\beta \int dt dz \left\{ |\partial n^l|^2 + (\bar{n} \partial_k n)^2 \right\}. \quad (2.23)$$

Here β is the two-dimensional coupling constant which is obtained from an integral over the profile functions of the quark and gauge fields over the transverse plane. Using the first-order differential equations (2.16) one can show that the integral is in fact a total derivative, and thus, determined by the boundary conditions (2.17). This yields

$$\beta = \frac{2\pi}{g_2^2}. \quad (2.24)$$

In quantum theory both coupling constants entering this equation run, and so one has to specify the scale at which the above relation holds. It is natural to set the scale of Eq. (2.24) to the cut-off scale of world-sheet dynamics, which is given by the inverse thickness of the string $g_2\sqrt{\xi}$.

Below $g_2\sqrt{\xi}$ the four-dimensional gauge couplings do not run due to the breaking of the gauge symmetry. The two-dimensional coupling starts logarithmic run below the cut-off scale,

$$4\pi\beta = N \ln \left(\frac{E}{\Lambda_{\text{CP}(N-1)}} \right). \quad (2.25)$$

By itself the $\text{CP}(N-1)$ theory is asymptotically free [18].

In the limit of large μ , the bulk theory becomes $\mathcal{N} = 1$ SQCD with its own scale (2.14). Using Eqs. (2.24) and (2.25), one can find the relation between the scales of the world-sheet and bulk theories [14],

$$\Lambda_{\text{CP}(N-1)} = \frac{\Lambda_{\mathcal{N}=1}^2}{g_2 \sqrt{\xi}}, \quad (2.26)$$

where the coupling constant is determined by the scale $\Lambda_{\mathcal{N}=1}$.

For a 1/2-BPS string, Eq. (2.23) gives only half of the world-sheet action, *i.e.* the bosonic part. The fermionic part of the theory is related to the bosonic one by

supersymmetry. For the string in the $\mathcal{N} = 2$ microscopic theory at hand, world-sheet dynamics is given by $\mathcal{N} = (2, 2)$ CP($N - 1$) sigma model,

$$\begin{aligned} \mathcal{S}_{1+1}^{(2,2)} = 2\beta \int d^2x \Bigg(& |\partial_k n|^2 + (\bar{n}\partial_k n)^2 + \bar{\xi}_L i\partial_R \xi_L + \bar{\xi}_R i\partial_L \xi_R \\ & - i(\bar{n}\partial_R n) \bar{\xi}_L \xi_L - i(\bar{n}\partial_L n) \bar{\xi}_R \xi_R \\ & + \bar{\xi}_L \xi_R \bar{\xi}_R \xi_L - \bar{\xi}_R \xi_R \bar{\xi}_L \xi_L \Bigg), \end{aligned} \quad (2.27)$$

where ξ^i is the two-dimensional fermionic superpartner of the orientational moduli n^l . The translational sector is completely decoupled from dynamics encoded in Eq. (2.27).

When $\mathcal{N} = 2$ supersymmetry is broken, the string internal dynamics is altered, and, as was shown in [10], the world sheet theory is given by $\mathcal{N} = (0, 2)$ CP($N - 1$) \times C. This theory has one dimensionless coupling $\tilde{\gamma}$ ¹ [11, 12], which is determined by the measure of supersymmetry breaking. In two-dimensional theory, this parameter sets the strength of the coupling of the translational sector to the orientational one,

$$\begin{aligned} S_{1+1}^{(0,2)} = 2\beta \int d^2x \Bigg(& \bar{\zeta}_R i\partial_L \zeta_R + \dots \\ & + |\partial n|^2 + (\bar{n}\partial_k n)^2 + \bar{\xi}_R i\partial_L \xi_R + \bar{\xi}_L i\partial_R \xi_L \\ & - i(\bar{n}\partial_L n) \bar{\xi}_R \xi_R - i(\bar{n}\partial_R n) \bar{\xi}_L \xi_L \\ & + \tilde{\gamma}(i\partial_L \bar{n})\xi_R \zeta_R + \bar{\tilde{\gamma}}\bar{\xi}_R(i\partial_L n)\bar{\zeta}_R + |\tilde{\gamma}|^2 \bar{\xi}_L \xi_L \bar{\zeta}_R \zeta_R \\ & + (1 - |\tilde{\gamma}|^2) \bar{\xi}_L \xi_R \bar{\xi}_R \xi_L - \bar{\xi}_L \xi_L \bar{\xi}_R \xi_R \Bigg). \end{aligned} \quad (2.28)$$

The ellipses denote the left-handed part of the translational sector, which stays decoupled.

The world sheet theory (2.28) can be obtained from the gauged formulation of CP($N - 1$) [19]. Details of the derivation of the CP($N - 1$) \times C action are given in [11, 12]. In this formulation, the most natural parameter of the theory δ arises as a constant in the quadratic deformation of the superpotential,

$$\mathcal{W}_{1+1} = \frac{1}{2} \delta \Sigma^2,$$

¹In this paper the heterotic deformation parameter $\tilde{\gamma}$ is related to the analogous parameter γ introduced in [11] as

$$\tilde{\gamma} = \sqrt{2/\beta} \gamma.$$

where Σ is a chiral superfield, a part of the gauge supermultiplet. The parameter δ is related to $\tilde{\gamma}$ via

$$\tilde{\gamma} = \frac{\sqrt{2}\delta}{\sqrt{1+2|\delta|^2}}.$$

The gauged formulation has a somewhat more direct physical interpretation than the representation (2.28), as the quantum behavior of the system is more directly seen in this picture [13]. In this sense, δ is also a more physical parameter, *e.g.* $\delta \rightarrow \infty$ supposedly corresponds to a conformal phase of the world sheet theory.

For a non-Abelian string in $\mathcal{N} = 2$ SQCD broken down to $\mathcal{N} = 1$ by soft mass terms μ_1 and μ_2 (see Eq. (2.1)), the relation between $\tilde{\gamma}$ and μ was found to be logarithmic [11, 12]

$$\delta = \text{const} \cdot \sqrt{\ln \frac{g_2^2 \mu}{m_W}} \quad (2.29)$$

for large μ . The large logarithm is associated with the emergence of the Higgs branch with light particles in $\mathcal{N} = 1$ SQCD in the limit of heavy adjoint superfields.

When the M field is present, the Higgs branch does not develop in the large- μ limit, and the adjoint fields are safely integrated out without disrupting the $\mathcal{N} = (0, 2)$ $\text{CP}(N-1)$ theory. The two-dimensional parameter $\tilde{\gamma}$ is then determined by the microscopic parameter h of the M model. Below we find the fermionic zero modes in the vortex background in the M model, and use them to obtain the relation between these two parameters.

3 Derivation of the Fermionic Part of the World-sheet Theory

The fermionic part of the M model (2.13) is²

$$\begin{aligned}
\mathcal{L}_{M\text{-model}}^{\text{ferm}} &= \frac{2i}{g_2^2} \text{Tr} \bar{\lambda}^{\text{SU}(N)} \overline{\mathcal{P}} \lambda^{\text{SU}(N)} + \frac{4i}{g_1^2} \bar{\lambda}^{\text{U}(1)} \overline{\mathcal{P}} \lambda^{\text{U}(1)} \\
&+ \text{Tr} i \bar{\psi} \overline{\mathcal{P}} \psi + \text{Tr} i \tilde{\psi} \overline{\mathcal{P}} \tilde{\psi} + \frac{2i}{h} \text{Tr} \overline{\vartheta} \mathcal{P} \vartheta \\
&+ i\sqrt{2} \text{Tr} \left(\bar{q} \lambda^{\text{U}(1)} \psi - \tilde{\psi} \lambda^{\text{U}(1)} \bar{q} + \overline{\psi} \lambda^{\text{U}(1)} q - \tilde{q} \lambda^{\text{U}(1)} \overline{\tilde{\psi}} \right) \\
&+ i\sqrt{2} \text{Tr} \left(\bar{q} \lambda^{\text{SU}(N)} \psi - \tilde{\psi} \lambda^{\text{SU}(N)} \bar{q} + \overline{\psi} \lambda^{\text{SU}(N)} q - \tilde{q} \lambda^{\text{SU}(N)} \overline{\tilde{\psi}} \right) \\
&+ i \text{Tr} \left(\tilde{q} \psi \vartheta + \tilde{\psi} q \vartheta + \overline{\psi} \tilde{q} \vartheta + \overline{q} \tilde{\psi} \vartheta \right) \\
&+ i \text{Tr} \left(\tilde{\psi} \psi M + \overline{\psi} \tilde{\psi} M \right),
\end{aligned} \tag{3.1}$$

where λ^α 's are fermionic $\mathcal{N} = 1$ superpartners of gauge fields, while $\psi^\alpha, \tilde{\psi}_{\dot{\alpha}}$ are matter fermions, $\alpha, \dot{\alpha} = 1, 2$ are their spinor indices. We use the decomposition

$$\vartheta_B^A = \frac{1}{2} \delta_B^A \vartheta^0 + (T^a)_B^A \vartheta^a \equiv \frac{1}{2} \delta_B^A \vartheta^0 + (\vartheta^N)_B^A$$

for the fermionic superpartner of the M field. We need to find the fermionic zero modes in the background of the vortex string (2.20).

In [14] an index theorem was derived, which shows that this theory possesses $4 + 4(N - 1)$ zero modes. The first four correspond to the fermionic superpartners ζ of the bosonic translational moduli x_0^1 and x_0^2 of the world-sheet theory. The other $4(N - 1)$ are the superorientational modes which are associated with the fermionic superpartners ξ^l of the orientational moduli n^l .

Since the M model possesses $1/2$ of supersymmetry of the original $\mathcal{N} = 2$ theory, one can utilize it in order to find one half of the fermionic zero modes — the ones that are associated with the left-handed fermions of the string world sheet. These supertransformations are identical to those of the original $\mathcal{N} = 2$ theory, and the corresponding zero modes were calculated in [12].

²We denote the fermionic superpartner of the M field as ϑ , in contrast to [14], where ζ was used. Here ζ is reserved for the world-sheet supertranslational variable.

We have, for the supertranslational modes,

$$\begin{aligned}
\bar{\psi}_2 &= -2\sqrt{2} \frac{x_1 + ix_2}{Nr^2} \left(\frac{1}{N} \phi_1(f + (N-1)f_N) + \frac{N-1}{N} \phi_2(f - f_N) \right. \\
&\quad \left. + (n\bar{n} - 1/N) \left\{ \phi_1(f + (N-1)f_N) - \phi_2(f - f_N) \right\} \right) \zeta_L, \\
\lambda^{1 \text{ U}(1)} &= -\frac{ig_1^2}{2} \left((N-1)\phi_2^2 + \phi_1^2 - N\xi \right) \zeta_L, \\
\lambda^{1 \text{ SU}(N)} &= -ig_2^2 (n\bar{n} - 1/N) \left(\phi_1^2 - \phi_2^2 \right) \zeta_L.
\end{aligned} \tag{3.2}$$

At the same time, for the superorientational modes we have

$$\begin{aligned}
\bar{\psi}_{2Ak} &= \frac{\phi_1^2 - \phi_2^2}{\phi_2} \cdot n\bar{\xi}_L, \\
\lambda^{1 \text{ SU}(N)} &= i\sqrt{2} \frac{x^1 - ix^2}{r^2} \frac{\phi_1}{\phi_2} f_N \cdot n\bar{\xi}_L.
\end{aligned} \tag{3.3}$$

Note that in Eqs. (3.2) and (3.3) we listed only nonvanishing components, which are (by definition) proportional to the left-handed fermions. As shown in [14], the M model has a $\text{U}(1)_{\tilde{R}}$ symmetry, under which these components have the charge +1. There must exist other, right-handed zero modes which are positively charged under the $\text{U}(1)_{\tilde{R}}$ symmetry as well. Although we will not need explicit expressions for the zero modes (3.2), (3.3) in what follows, they are helpful in finding a good *ansatz* for the right-handed zero modes. It is rather obvious that by substituting the modes (3.2) and (3.3) into the kinetic terms of Eq. (3.1) one recovers the left-handed kinetic part of the $\text{CP}(N-1) \times \text{C}$ model (2.28).

To obtain the right-handed zero modes one generally needs to solve the Dirac equations. We follow the approach of [14] where the parameter h was tuned to be small (but non-zero),

$$0 < h \ll g_2^2, \tag{3.4}$$

allowing to find the solution analytically. We deal with supertranslational and superorientational modes in turn.

First we make a guess on what fields should participate in the right-handed modes. From the mass-deformed $\mathcal{N} = 2$ case [12] we know that they would involve the fields $\bar{\widetilde{\psi}}_1$ and λ^{22} but the latter field is not present in our M model as it was integrated out. We infer then that the correct set of fermions which constitute the right-handed modes are $\bar{\widetilde{\psi}}_1$, ϑ^0 and ϑ^a , *i.e.* those which have the $\text{U}(1)_{\tilde{R}}$ charges +1 and couple to $\bar{\widetilde{\psi}}_1$. The fermions λ^1 and $\bar{\psi}_2$ decouple from this set completely. They are given by (3.2), (3.3).

3.1 Supertranslational Zero Modes

The Dirac equations for the $\widetilde{\psi}_i$ and ϑ are

$$\begin{aligned} i \nabla^{2i} \widetilde{\psi}_i + i q \left(\frac{1}{2} \vartheta_0 + \vartheta^N \right) &= 0, \\ \frac{i}{h} \bar{\partial}_{i2} (\vartheta^N)^2 + \frac{i}{2} \text{Traceless} \left\{ q \widetilde{\psi}_i \right\} &= 0, \\ \frac{i}{h} \bar{\partial}_{i2} \vartheta^0 + \frac{i}{N} \text{Tr} \left(q \widetilde{\psi}_i \right) &= 0. \end{aligned}$$

For constructing an appropriate *ansatz* for the solution we split the trace and traceless components of the fields, which we mark by superscripts 0 and N , respectively, attributing different profile functions to them. One has a freedom of placing a factor of $(x^1 \pm ix^2)/r$ in these components, which gives two possibilities for the *ansätze* for the zero modes. Let us call ζ_R and $\bar{\zeta}_R$ the corresponding world-sheet fermions; then

$$\begin{aligned} \widetilde{\psi}_i &= \frac{1}{2} \left(\chi_{\text{tr}}^0 + N(n\bar{n} - 1/N) \chi_{\text{tr}}^N \right) \zeta_R + \\ &\quad + \frac{1}{2} \frac{x^1 - ix^2}{r} \left(\psi_{\text{tr}}^0 + N(n\bar{n} - 1/N) \psi_{\text{tr}}^N \right) \bar{\zeta}_R, \\ (\vartheta^0)^2 &= \frac{x^1 + ix^2}{r} \rho_{\text{tr}}^0(r) \cdot \zeta_R + \vartheta_{\text{tr}}^0(r) \cdot \bar{\zeta}_R, \\ (\vartheta^N)^2 &= \frac{x^1 + ix^2}{r} \rho_{\text{tr}}^N(r)(n\bar{n} - 1/N) \cdot \zeta_R + \vartheta_{\text{tr}}^N(r) \cdot (n\bar{n} - 1/N) \cdot \bar{\zeta}_R. \end{aligned}$$

Here the superscript “tr” is used to denote the profile functions of the supertranslational modes, versus the superorientational modes to appear later. Substituting this into the Dirac equations, one obtains the equations for the profile functions $\chi_{\text{tr}}^{0,N}$ and

$\rho_{\text{tr}}^{0,N}$,

$$\begin{aligned}
\partial_r \chi_{\text{tr}}^0 &- \frac{1}{Nr} \left(f \chi_{\text{tr}}^0 + (N-1) f_N \chi_{\text{tr}}^N \right) \\
&+ i \frac{\phi_1 + (N-1) \phi_2}{N} \rho_{\text{tr}}^0 + 2i \frac{N-1}{N^2} (\phi_1 - \phi_2) \rho_{\text{tr}}^N = 0, \\
\partial_r \chi_{\text{tr}}^N &- \frac{1}{Nr} \left(f \chi_{\text{tr}}^N + f_N \chi_{\text{tr}}^0 + (N-2) f_N \chi_{\text{tr}}^N \right) \\
&+ i \frac{\phi_1 - \phi_2}{N} \rho_{\text{tr}}^0 + 2i \frac{(N-1)\phi_1 + \phi_2}{N^2} \rho_{\text{tr}}^N = 0, \\
&- \frac{1}{h} \left\{ \partial_r + \frac{1}{r} \right\} \rho_{\text{tr}}^N + \frac{i}{4} \left[(\phi_1 + (N-1) \phi_2) \chi_{\text{tr}}^N \right. \\
&\quad \left. + (\phi_1 - \phi_2) \chi_{\text{tr}}^0 + (N-2) (\phi_1 - \phi_2) \chi_{\text{tr}}^N \right] = 0, \\
&- \frac{1}{h} \left\{ \partial_r + \frac{1}{r} \right\} \rho_{\text{tr}}^0 + \frac{i}{2N} \left[(\phi_1 + (N-1) \phi_2) \chi_{\text{tr}}^0 \right. \\
&\quad \left. + (N-1) (\phi_1 - \phi_2) \chi_{\text{tr}}^N \right] = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\left\{ \partial_r + \frac{1}{r} \right\} \psi_{\text{tr}}^0 - \frac{1}{Nr} \left(f \psi_{\text{tr}}^0 + (N-1) f_N \psi_{\text{tr}}^N \right) \\
&+ i \frac{\phi_1 + (N-1) \phi_2}{N} \vartheta_{\text{tr}}^0 + 2i \frac{N-1}{N^2} (\phi_1 - \phi_2) \vartheta_{\text{tr}}^N = 0, \\
&\left\{ \partial_r + \frac{1}{r} \right\} \psi_{\text{tr}}^N - \frac{1}{Nr} \left(f \psi_{\text{tr}}^N + f_N \psi_{\text{tr}}^0 + (N-2) f_N \psi_{\text{tr}}^N \right) \\
&+ i \frac{\phi_1 - \phi_2}{N} \vartheta_{\text{tr}}^0 + 2i \frac{(N-1)\phi_1 + \phi_2}{N^2} \vartheta_{\text{tr}}^N = 0, \quad (3.5) \\
&- \frac{1}{h} \partial_r \vartheta_{\text{tr}}^N + \frac{i}{4} \left[(\phi_1 + (N-1) \phi_2) \psi_{\text{tr}}^N \right. \\
&\quad \left. + (\phi_1 - \phi_2) \psi_{\text{tr}}^0 + (N-2) (\phi_1 - \phi_2) \psi_{\text{tr}}^N \right] = 0, \\
&- \frac{1}{h} \partial_r \vartheta_{\text{tr}}^0 + \frac{i}{2N} \left[(\phi_1 + (N-1) \phi_2) \psi_{\text{tr}}^0 \right. \\
&\quad \left. + (N-1) (\phi_1 - \phi_2) \psi_{\text{tr}}^N \right] = 0
\end{aligned}$$

for the profile functions $\psi_{\text{tr}}^{U,N}$ and $\vartheta_{\text{tr}}^{0,N}$. Only the second set of equations turns out to yield solutions finite at $r \rightarrow 0$. So we drop χ_{tr} and ρ_{tr} , and accept

$$\begin{aligned}
(\vartheta^0)^2 &= \vartheta_{\text{tr}}^0(r) \cdot \zeta_R, \\
(\vartheta^N)^2 &= \vartheta_{\text{tr}}^N(r) (n\bar{n} - 1/N) \cdot \zeta_R, \\
\bar{\psi}_1 &= \frac{1}{2} \frac{x^1 - ix^2}{r} \left(\psi_{\text{tr}}^0 + N (n\bar{n} - 1/N) \psi_{\text{tr}}^N \right) \zeta_R
\end{aligned} \quad (3.6)$$

for the zero modes. Following [14, 12] we solve Eqs. (3.5) in the limits of large r , *i.e.* $r \gg 1/(g_2\sqrt{\xi})$ and intermediate r , *i.e.* $r \lesssim 1/(g_2\sqrt{\xi})$. The idea is that if h is small (see (3.4)) we have the matter fields in our theory which are much lighter than the gauge bosons, see (2.11), (2.12). Although the light scalar fields vanish on the string solution, the presence of light fermion fields affects the fermionic sector of the theory [20, 11]. In particular, the string fermion zero modes have a two-layer structure in the plane orthogonal to the string axis: a core of the size of the inverse gauge boson mass plus long-range “tails” formed by light fermions. Completely analogously to calculations in [14, 12], in the limit of small h , we find in the large- r domain,

$$\begin{aligned}\vartheta_{\text{tr}}^0 &= -C i \frac{m_0^2}{\sqrt{\xi}} K_0(m_0 r), & \vartheta_{\text{tr}}^N &= -C i \frac{N}{2} \frac{m_0^2}{\sqrt{\xi}} K_0(m_0 r), \\ \psi_{\text{tr}}^0 &= -C \partial_r K_0(m_0 r), & \psi_{\text{tr}}^N &= -C \partial_r K_0(m_0 r),\end{aligned}$$

where $K_0(z)$ is the McDonald function, while at intermediate r we get

$$\begin{aligned}\psi_{\text{tr}}^0 &= \psi_{\text{tr}}^N = \frac{C}{\sqrt{\xi}} \frac{\phi_1}{r}, \\ \vartheta_{\text{tr}}^0 &\simeq -C i \frac{m_0^2}{\sqrt{\xi}} \ln \frac{m_W}{m_0}, \\ \vartheta_{\text{tr}}^N &\simeq -C i \frac{N}{2} \frac{m_0^2}{\sqrt{\xi}} \ln \frac{m_W}{m_0},\end{aligned}$$

where

$$m_0 \equiv \sqrt{\frac{h}{2}} \xi.$$

The arbitrary constant C is common for all profile functions, and we can safely put $C = 1$.

3.2 Superorientational Zero Modes

Orientational fermion zero modes in the M model with the gauge group $U(2)$ were calculated in [14]. Here we generalize these results to the theory with the $U(N)$ gauge group. Now the trace component of the ϑ field is not involved; therefore, we need to deal only with two Dirac equations,

$$\begin{aligned}i \nabla^{2i} \widetilde{\psi}_i + i q \vartheta^N &= 0, \\ \frac{i}{h} \bar{\partial}_{i2} (\vartheta^N)^2 + \frac{i}{2} \text{Traceless} \left\{ q \widetilde{\psi}_i \right\} &= 0.\end{aligned}\tag{3.7}$$

The form of the zero modes (3.3) prompts us that the right-handed modes may be proportional to $n \bar{\xi}_R$, or $\xi_R \bar{n}$, which gives two possibilities for the *ansatz*. One also

has the freedom of putting the factor of $(x^1 \pm ix^2)/r$ in either $\widetilde{\psi}_i$ or ϑ^N . Overall, we write the following four *ansätze*:

$$\begin{aligned}
(\vartheta^N)^2 &= 2 \vartheta_{\text{or}}(r) \cdot n \bar{\xi}_R + 2 \eta_{\text{or}}(r) \cdot \xi_R \bar{n} \\
&\quad + 2 \frac{x^1 + ix^2}{r} \kappa_{\text{or}}(r) \cdot n \bar{\xi}_R + 2 \frac{x^1 + ix^2}{r} \sigma_{\text{or}}(r) \cdot \xi_R \bar{n}, \\
\widetilde{\psi}_i &= 2 \frac{x^1 - ix^2}{r} \psi_{\text{or}}(r) \cdot n \bar{\xi}_R + 2 \frac{x^1 - ix^2}{r} \chi_{\text{or}}(r) \cdot \xi_R \bar{n} \\
&\quad + 2 v_{\text{or}}(r) \cdot n \bar{\xi}_R + 2 \omega_{\text{or}}(r) \cdot \xi_R \bar{n},
\end{aligned}$$

where the subscript “or” is added to distinguish the profile functions from those of the translational modes. Plugging these into the Dirac equations (3.7), we have eight equations for the profile functions

$$\begin{aligned}
& - \partial_r \vartheta_{\text{or}}(r) + \frac{ih}{2} \phi_1(r) \psi_{\text{or}}(r) = 0, \\
& \left\{ \partial_r + \frac{1}{r} \right\} \psi_{\text{or}}(r) - \frac{1}{Nr} \left(f + (N-1) f_N \right) \psi_{\text{or}}(r) + i \phi_1(r) \vartheta_{\text{or}}(r) = 0, \\
& - \left\{ \partial_r + \frac{1}{r} \right\} \eta_{\text{or}}(r) + \frac{ih}{2} \phi_1(r) \chi_{\text{or}}(r) = 0, \\
& \partial_r \chi_{\text{or}}(r) - \frac{1}{Nr} \left(f + (N-1) f_N \right) \chi_{\text{or}}(r) + i \phi_1(r) \eta_{\text{or}}(r) = 0, \tag{3.8} \\
& - \partial_r \kappa_{\text{or}}(r) + \frac{ih}{2} \phi_2(r) v_{\text{or}}(r) = 0, \\
& \left\{ \partial_r + \frac{1}{r} \right\} v_{\text{or}}(r) - \frac{1}{Nr} \left(f - f_N \right) v_{\text{or}}(r) + i \phi_2(r) \kappa_{\text{or}}(r) = 0, \\
& - \left\{ \partial_r + \frac{1}{r} \right\} \sigma_{\text{or}}(r) + \frac{ih}{2} \phi_2(r) \omega_{\text{or}}(r) = 0, \\
& \partial_r \omega_{\text{or}}(r) - \frac{1}{Nr} \left(f - f_N \right) \omega_{\text{or}}(r) + i \phi_2(r) \sigma_{\text{or}}(r) = 0.
\end{aligned}$$

From this set of four pairs of equations only the first pair yields nonsingular profile functions for the zero modes. Thus we have

$$\begin{aligned}
(\vartheta^N)^2 &= 2 \vartheta_{\text{or}}(r) \cdot n \bar{\xi}_R, \\
\widetilde{\psi}_i &= 2 \frac{x^1 - ix^2}{r} \psi_{\text{or}}(r) \cdot n \bar{\xi}_R. \tag{3.9}
\end{aligned}$$

Again, we solve the equations (3.8) separately in the domain of large r and intermediate r , assuming h to be small. Parallelizing [14, 12], we get for large r

$$\begin{aligned}\vartheta_{\text{or}}(r) &= -\frac{ih\sqrt{\xi}}{2} K_0(m_0 r), \\ \psi_{\text{or}}(r) &= -\partial_r K_0(m_0 r),\end{aligned}\tag{3.10}$$

while in the intermediate- r domain the profile functions take the form

$$\begin{aligned}\vartheta_{\text{or}}(r) &\simeq -\frac{ih\sqrt{\xi}}{2} \ln \frac{m_W}{m_0}, \\ \psi_{\text{or}}(r) &\simeq \frac{\phi_1}{\sqrt{\xi} r}.\end{aligned}\tag{3.11}$$

Relative normalization of equations (3.10) and (3.11) has been taken care of to ensure agreement between the two domains. The overall normalization, similarly to the supertranslational case, is given by a common arbitrary constant which we have put to one.

We observe that the right-handed zero modes exhibit the long-range $1/r$ behavior similar to that observed in the $\mathcal{N} = 2$ theory deformed solely by the $\mu_{1,2}$ parameters [14, 11]. This is expected, as in the limit $h \rightarrow 0$ the theory re-acquires the Higgs branch, and the associated massless modes. We have no need, however, of taking this limit; we chose h to be small, see Eq. (3.4), only for the purpose of making analytical computations simpler. At the same time h can be (and is) treated as a fixed parameter. Decoupling of the adjoint fields does not depend on the value of h .

3.3 Bifermionic Coupling

The easiest way to obtain the coupling constant $\tilde{\gamma}$ of the heterotic $\text{CP}(N-1)$ model is to calculate the strength of the coupling of the supertranslational and superorientational modes induced on the string world sheet

$$\mathcal{L}_{\text{eff}}^{\mathcal{N}=(0,2)} \supset 2\beta \cdot I_{\zeta\xi} (i\partial_L \bar{n} \xi_R \zeta_R + \bar{\xi}_R i\partial_L n \bar{\zeta}_R),\tag{3.12}$$

where we separate the factor 2β for convenience. It is natural to assume that $\tilde{\gamma}$ will be real, since the deformation h is.

To be able to compare this coupling constant to $\tilde{\gamma}$ in Eq. (2.28) one has to normalize the participating fermions. We define the normalization integrals I_ζ and I_ξ for the fermions ζ and ξ as

$$\mathcal{L}_{\text{eff}}^{\mathcal{N}=(0,2)} \supset 2\beta \cdot (I_\zeta \bar{\zeta}_R i\partial_L \zeta_R + I_\xi \bar{\xi}_R i\partial_L \xi_R).\tag{3.13}$$

Substituting the expressions (3.6) and (3.9) for the zero modes into the kinetic terms of the microscopic theory (3.1) one obtains

$$I_\zeta = \frac{N}{2\xi} \frac{m_W^2}{4} \int r dr \left\{ -\frac{2}{h} \left\{ (\vartheta_{\text{tr}}^0)^2 + 4 \frac{N-1}{N^2} (\vartheta_{\text{tr}}^N)^2 \right\} \right. \\ \left. + (\psi_{\text{tr}}^0)^2 + (N-1) (\psi_{\text{tr}}^N)^2 \right\},$$

$$I_\xi = 2 g_2^2 \int r dr \left(-\frac{2}{h} (\vartheta_{\text{or}}(r))^2 + (\psi_{\text{or}}(r))^2 \right).$$

From a similar procedure one extracts the expression for the bifermionic coupling

$$I_{\zeta\xi} = \frac{g_2^2}{2} \int r dr \left(-\frac{4}{h} \vartheta_{\text{or}} \vartheta_{\text{tr}}^N + N \psi_{\text{or}} \psi_{\text{tr}}^N + \rho \psi_{\text{or}} (\psi_{\text{tr}}^0 - \psi_{\text{tr}}^N) \right).$$

Substituting here the profile functions of the zero modes, in particular, their $1/r$ -tails, we obtain with logarithmic accuracy

$$I_\zeta = N^2 \frac{g_2^2}{8} \ln \frac{m_W}{m_0},$$

$$I_\xi = 2 g_2^2 \ln \frac{m_W}{m_0},$$

$$I_{\zeta\xi} = N \frac{g_2^2}{2} \ln \frac{m_W}{m_0}.$$

The logarithms come from the domain $1/m_W \ll r \ll 1/m_0$ of the zero mode profile functions.

Normalizing the world-sheet fermions using the above integrals, one arrives at the answer for the world-sheet coupling $\tilde{\gamma}$ up to a contribution suppressed by the inverse large logarithms. Parametrically, the logarithms under consideration can be written as

$$\ln \frac{m_W}{m_0} \simeq \ln \frac{g_2}{\sqrt{h}} = \frac{1}{2} \ln \frac{g_2^2}{h},$$

where h is small. Overall our result takes the form

$$\tilde{\gamma} = \frac{\sqrt{2}\delta}{\sqrt{1 + 2|\delta|^2}} = \frac{I_{\zeta\xi}}{\sqrt{I_\zeta I_\xi}} = 1 + O\left(\frac{1}{\ln g_2^2/h}\right)$$

(remind that in this paper $\tilde{\gamma}$ is related to γ of [11] as $\tilde{\gamma} = \sqrt{2/\beta} \gamma$). Therefore,

$$\delta = \text{const} \cdot \sqrt{\ln g_2^2/h}. \quad (3.14)$$

This relation is, of course, quite analogous to that for the deformation parameter δ in the heterotic string scenario of Refs. [11, 12]. The difference, however, is that δ does not go all the way to infinity in the limit $\mu \rightarrow \infty$; hence the $\text{CP}(N-1) \times \text{C}$ model gives a reliable description of the string world sheet in this limit.

4 Confined monopoles

Since the bulk theory quarks are in the Higgs phase, the monopoles are confined. It was shown in [4, 3, 5] that when we introduce a nonvanishing FI parameter ξ in $U(N)$ $\mathcal{N} = 2$ SQCD, the 't Hooft–Polyakov monopoles of the $SU(N)$ subgroup become confined on the string — they become string junctions of two elementary non-Abelian strings. Each string of the bulk theory corresponds to a particular vacuum of the world-sheet theory. In particular, $\mathcal{N} = (2, 2)$ supersymmetric $CP(N - 1)$ model on the string world sheet has N degenerate vacua and kinks interpolating between different vacua. These kinks are interpreted as confined monopoles of the bulk theory [4, 3, 5]. In the limit of massless quarks these monopoles become truly non-Abelian. They no longer carry average magnetic flux since

$$\langle n^l \rangle = 0 \tag{4.1}$$

in the strong coupling limit of the $CP(N - 1)$ model. Still these monopoles (= kinks) are stabilized by quantum effects in the $CP(N - 1)$ model. They acquire mass and an inverse size of the order of $\Lambda_{CP(N-1)}$. They are described by fields n^l and form the fundamental representation of the $SU(N)_{C+F}$ group [21].

Now what happens with these monopoles when we introduce $\mathcal{N} = 2$ supersymmetry breaking parameter μ and tend it to infinity converting the microscopic theory into $\mathcal{N} = 1$ SQCD? (We assume that $\mu_1 \sim \mu_2 \equiv \mu$). Note that $\mathcal{N} = 1$ SQCD has no adjoint fields at all (they completely decouple), so in no way the monopoles can be seen quasiclassically. Moreover, no breaking of the gauge group to an Abelian subgroup occurs in this theory; therefore, the monopoles (if exist) should be truly non-Abelian.

This question was addressed in [11], where it was noted that $\mathcal{N} = 1$ SQCD develops a Higgs branch in the limit $\mu \rightarrow \infty$, and therefore the fate of the confined monopoles can be traced only up to a finite value of μ , see Eq. (1.1). On the other hand, in the M model there is no Higgs branch in the limit $\mu \rightarrow \infty$ and the presence of confined monopoles was traced all the way to $\mu = \infty$ [14]. In this limit we get a remarkable result: although the adjoint fields are eliminated from our theory and monopoles cannot be seen in any semiclassical description, our analysis shows that confined non-Abelian monopoles still exist in the theory (2.13). They are seen as $CP(N - 1)$ -model kinks in the effective world-sheet theory on the non-Abelian string.

The only loophole in the above argument is that the fermionic sector of the world-sheet theory was not studied in [14]. In fact, it was not clear, whether the world-sheet theory has N strictly degenerate vacua and kinks interpolating between them (to be interpreted as confined monopoles). Say, if N vacua were split (as it happens in the nonsupersymmetric case [17]) a monopole and antimonopole attached to the string

would come close to each other to form a meson-like configuration, see the review [8] for details. If in the large- μ limit the splittings were large, the binding inside these mesons could become stronger and individual monopoles would not be seen. This effect corresponds to the kink confinement in two-dimensional nonsupersymmetric $\text{CP}(N-1)$ model [21].

In this paper we completed the proof of the presence of confined non-Abelian monopoles in the M model in the limit $\mu \rightarrow \infty$. By confined we mean confined on the string but unconfined along the string. Above we demonstrated that the world-sheet theory on the non-Abelian string is heterotic $\mathcal{N} = (0, 2)$ supersymmetric $\text{CP}(N-1)$ model. We derived Eq. (3.14) which relates the deformation parameter of the world sheet theory to parameters of the bulk theory in the large μ limit. In particular, it shows that δ goes to a constant at large μ .

Physics of the heterotic $\mathcal{N} = (0, 2)$ supersymmetric $\text{CP}(N-1)$ model was studied in the large- N approximation in [13]. In this paper it was shown that supersymmetry is spontaneously broken (see also [22]). Still the Z_{2N} discrete symmetry present in the model is spontaneously broken down to Z_2 , and the model has N strictly degenerate vacua. This ensures the presence of kinks, interpolating between these vacua. These kinks are confined non-Abelian monopoles of the bulk theory.

The kink masses were calculated in [13] in the large- N approximation. As was already mentioned, the kinks in the strong coupling regime are described by fields n^l [21]. Due to spontaneous supersymmetry breaking the n^l boson masses and those of fermions ξ are different. Namely [13],

$$m_n = \Lambda_{\text{CP}(N-1)}, \quad m_\xi = \Lambda_{\text{CP}(N-1)} \exp\left(-\frac{8\pi\beta}{N} |\delta|^2\right), \quad (4.2)$$

where δ is given in Eq. (3.14), while the coupling constant

$$\beta = \frac{N}{4\pi} \ln\left(\frac{m_W}{\Lambda_{\text{SU}(N)}}\right). \quad (4.3)$$

As we see, these expressions give finite nonvanishing masses for bosonic and fermionic components of the non-Abelian monopoles confined to the string. The masses are well defined in the limit $\mu \rightarrow \infty$ since the parameter δ stays finite in this limit.

5 Conclusions

This paper concludes the program started in [14], namely direct and explicit derivation of the world-sheet theory for non-Abelian strings in the M model starting from the bulk theory with $\mathcal{N} = 2$ supersymmetry broken down to $\mathcal{N} = 1$ by the mass terms

of the adjoint fields and the coupling of the quark fields to the M field. We demonstrated that the world sheet theory on the non-Abelian string is heterotic $\mathcal{N} = (0, 2)$ supersymmetric $CP(N-1)$ model. To this end we had to explicitly obtain all fermion zero modes in the limit of large μ . We related the deformation parameter δ of the world-sheet theory to parameters of the bulk theory and showed that δ does not depend on μ at large μ .

This completes the proof of the presence of non-Abelian monopoles confined to the non-Abelian strings in the M model. Note that at $\mu \rightarrow \infty$ the bulk theory does not have adjoint fields and monopoles cannot be seen in the quasiclassical approximation. We showed that they are still present in the theory and are seen as kinks in the heterotic $CP(N-1)$ model on the string. These kinks are stabilized by non-perturbative effects in two dimensions (e.g. two-dimensional instantons) in the limit when the color-flavor locked $SU(N)$ symmetry is attained in the bulk theory.

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